

# EIGENVALUE ESTIMATES FOR SUBMANIFOLDS OF $N \times \mathbb{R}$ WITH LOCALLY BOUNDED MEAN CURVATURE

G. PACELLI BESSA AND M. SILVANA COSTA

**ABSTRACT.** We give lower bounds for the fundamental tone of open sets in submanifolds with locally bounded mean curvature in  $N \times \mathbb{R}$ , where  $N$  is an  $n$ -dimensional complete Riemannian manifold with radial sectional curvature  $K_N \leq \kappa$ . When the immersion is minimal our estimates are sharp. We also show that cylindrically bounded minimal surfaces has positive fundamental tone.

## 1. INTRODUCTION

The fundamental tone  $\lambda^*(\Omega)$  of an open set  $\Omega$  in a smooth Riemannian manifold  $M$  is defined by

$$\lambda^*(\Omega) = \inf \left\{ \frac{\int_{\Omega} |\text{grad } f|^2}{\int_{\Omega} f^2}; f \in H_0^1(\Omega) \setminus \{0\} \right\}.$$

When  $\Omega = M$  is an open Riemannian manifold, the fundamental tone  $\lambda^*(M)$  coincides with the greatest lower bound  $\inf \Sigma$  of the spectrum  $\Sigma \subset [0, \infty)$  of the unique self-adjoint extension of the Laplacian  $\Delta$  acting on  $C_0^\infty(M)$  also denoted by  $\Delta$ . When  $\Omega$  is compact with piecewise smooth boundary  $\partial\Omega$  (possibly empty) then  $\lambda^*(\Omega)$  is the first eigenvalue  $\lambda_1(\Omega)$  of  $\Omega$  (Dirichlet boundary data if  $\partial\Omega \neq \emptyset$ ). A well studied problem in the geometry of the Laplacian is the relations between the first eigenvalue or the fundamental tone of open sets of Riemannian manifolds and the manifold's geometric invariants, see [2], [3], [8] and references therein. Another kind of problem is to give bounds for the the first eigenvalue (fundamental tone) of open sets of minimal submanifolds of Riemannian manifolds, see [4], [5], [7], [9], [10]. There has been recently an increasingly interest on the study of minimal surfaces (constant mean curvature) in product spaces  $N \times \mathbb{R}$ , with after the discovery of many beautiful examples in those spaces, see [14], [15]. This motivates us to study the fundamental tone of minimal submanifold of product spaces  $N \times \mathbb{R}$ . Our first result is the following theorem.

**Theorem 1.1.** *Let  $\varphi : M \hookrightarrow N \times \mathbb{R}$  be a complete minimal  $m$ -dimensional submanifold, where  $N$  has radial sectional curvature  $K(\gamma(t))(\gamma'(t), v) \leq \kappa$ ,  $v \in T_{\gamma(t)}N$ ,  $|v| = 1$ ,  $v \perp \partial t$ , along the geodesics  $\gamma(t)$  issuing from a point  $x_0 \in N$ . Let  $\Omega \subset \varphi^{-1}(B_N(x_0, r) \times \mathbb{R})$  be a connected component, where  $r < \min\{\text{inj}(x_0), \pi/2\sqrt{\kappa}\}$ ,  $(\pi/2\sqrt{\kappa} = \infty$  if  $\kappa \leq 0$ ). Then*

$$(1.1) \quad \lambda^*(\Omega) \geq \lambda_1(B_{\mathbb{N}^{m-1}(\kappa)}(r)).$$

*If  $\Omega$  is bounded then inequality (1.1) is strict. Here  $\mathbb{N}^{m-1}(\kappa)$  is the  $(m-1)$ -dimensional simply connected space form of constant sectional curvature  $\kappa$ .*

---

*Date:*

2000 *Mathematics Subject Classification.* Primary 53C40, 53C42; Secondary 58C40.

*Key words and phrases.* Fundamental tone estimates, minimal submanifolds, submanifolds with locally bounded mean curvature in  $N \times \mathbb{R}$ .

The first author was partially supported by a CNPq-grant and ICTP Associate Schemes.

The second author was partially supported by a CNPq-scholarship.

Theorem (1.1) can be viewed as a version of Theorem (1.10) of [5] for product spaces. There Bessa and Montenegro gave eigenvalue estimates for pre-images of geodesic balls in Riemannian manifolds with radial sectional curvature bounded above, here we give lower eigenvalue estimates for pre-images of cylinders in product spaces.

*Remark 1.2.* Inequality (1.1) is sharp. For if we let  $\varphi : \mathbb{H}^{m-1}(-1) \times \mathbb{R} \hookrightarrow \mathbb{H}^n(-1) \times \mathbb{R}$  be given by  $\varphi(x, t) = (i(x), t)$ , where  $i : \mathbb{H}^{m-1}(-1) \hookrightarrow \mathbb{H}^n(-1)$  is a totally geodesic embedding then for  $\Omega = \varphi^{-1}(B_{\mathbb{H}^n(-1)}(r) \times \mathbb{R}) = B_{\mathbb{H}^{m-1}(-1)}(r) \times \mathbb{R}$  we have

$$\lambda^*(\Omega) = \lambda_1(B_{\mathbb{H}^{m-1}(-1)}(r)).$$

**Corollary 1.3.** *Let  $\varphi : M \hookrightarrow \mathbb{R}^3$  be a complete minimal surface with  $\varphi(M) \subset B_{\mathbb{R}^2}(r) \times \mathbb{R}$ . Then*

$$(1.2) \quad \lambda^*(M) \geq \lambda_1(B_{\mathbb{R}^2}(r)) = \frac{c_0}{r^2},$$

where  $c_0$  is the first zero of the  $J_0$ -Bessel function.

**Question 1.4.** The only examples known (to the best of our knowledge) of complete surfaces in  $\mathbb{R}^3$  with positive fundamental tone are the Nadirashvilli bounded minimal surfaces [16] and Martin-Morales cylindrically bounded minimal surfaces [13]. Both Nadirashvilli and Martin-Morales minimal surfaces have at least two bounded coordinate functions. That was crucial in the proof that their fundamental tones were positive. That raises the question whether there are minimal surfaces in  $\mathbb{R}^3$  with at most one bounded coordinate function with positive fundamental tone. More specifically, has the Jorge-Xavier minimal surface inside the slab [12] positive fundamental tone?

A second purpose of this paper is study the fundamental tones of domains in submanifolds of  $N \times \mathbb{R}$  with locally bounded mean curvature. We need a stronger notion of *locally bounded mean curvature* than the considered in [4].

**Definition 1.5.** An isometric immersion  $\varphi : M \hookrightarrow W \times \mathbb{R}$  has locally bounded mean curvature  $|H|$  if for any  $p \in W$  and  $r > 0$ , the number

$$h(p, r) = \sup\{|H(x)|; x \in \varphi(M) \cap (B_W(p, r) \times \mathbb{R})\}$$

is finite. Here  $B_W(x_0, r)$  is the geodesic ball of radius  $r$  and center  $x_0$  in  $W$ .

Our second result is the following theorem.

**Theorem 1.6.** *Let  $\varphi : M \hookrightarrow N \times \mathbb{R}$  be a complete immersed  $m$ -submanifold of  $N \times \mathbb{R}$  with locally bounded mean curvature, where  $N$  has radial sectional curvature  $K \leq \kappa$ , along the geodesics issuing from a point  $x_0 \in N$ . Let  $\Omega(r) \subset \varphi^{-1}(B_N(p, r) \times \mathbb{R})$  be any connected component. Suppose  $r \leq \min\{\text{inj}_N(x_0), \pi/2\sqrt{\kappa}\}$ . In addition*

- If  $|h(x_0, r)| < \Lambda^2 < \infty$  let  $r \leq (C_\kappa/S_\kappa)^{-1}(\Lambda^2/(m-2))$ .
- If  $\lim_{r \rightarrow \infty} h(x_0, r) = \infty$  let  $r \leq (C_\kappa/S_\kappa)^{-1}(h(x_0, r_0)/(m-2))$ , where  $r_0$  is so that  $(m-2)\frac{C_\kappa}{S_\kappa}(r_0) - h(x_0, r_0) = 0$ .

In both cases we have

$$\lambda^*(\Omega(r)) \geq \left[ \frac{(m-2)\frac{C_\kappa}{S_\kappa}(r) - h(x_0, r)}{2} \right]^2 > 0.$$

**Corollary 1.7** (Bessa-Montenegro, [6]). *Let  $\varphi : M \hookrightarrow N \times \mathbb{R}$  be a compact immersed submanifold with mean curvature vector  $H$ . Let  $p_1 : N \times \mathbb{R} \rightarrow N$  be the projection on the first factor. Then the extrinsic radius of  $p_1(M)$  is given*

$$R_{p_1(M)} = \left( \frac{C_\kappa}{S_\kappa} \right)^{-1} \left( \sup_M |H| / (m-2) \right).$$

## 2. PRELIMINARIES

Let  $\varphi : M \hookrightarrow W$  be an isometric immersion, where  $M$  and  $W$  are complete Riemannian manifolds of dimension  $m$  and  $n$  respectively. Consider a smooth function  $g : W \rightarrow \mathbb{R}$  and the composition  $f = g \circ \varphi : M \rightarrow \mathbb{R}$ . Identifying  $X$  with  $d\varphi(X)$  we have at  $q \in M$  that the Hessian of  $f$  is given by

$$(2.1) \quad \text{Hess } f(q)(X, Y) = \text{Hess } g(\varphi(q))(X, Y) + \langle \text{grad } g, \alpha(X, Y) \rangle_{\varphi(q)}.$$

Taking the trace in (2.1), with respect to an orthonormal basis  $\{e_1, \dots, e_m\}$  for  $T_q M$ , we have the Laplacian of  $f$ ,

$$(2.2) \quad \Delta f(q) = \sum_{i=1}^m \text{Hess } g(\varphi(q))(e_i, e_i) + \langle \text{grad } g, \sum_{i=1}^m \alpha(e_i, e_i) \rangle.$$

The formulas (2.1) and (2.2) are well known in the literature, see [11]. For the proof of Theorems (1.1) and (1.6) we will need few preliminaries results. The first result we need is the Hessian Comparison Theorem, one can see [17] for a proof.

**Theorem 2.1** (Hessian Comparison Theorem). *Let  $W$  be a complete Riemannian manifold and let  $\rho$  be the distance function on  $W$  to  $x_0$ . Let  $\gamma$  be a minimizing geodesic starting at  $x_0$  and suppose that the radial sectional curvatures of  $W$  along  $\gamma$  is bounded above  $K_\gamma \leq \kappa$ . Then the Hessian of  $\rho$  at  $\gamma(t)$  satisfies*

$$(2.3) \quad \text{Hess } \rho(\gamma(t))(X, X) \geq \frac{C_\kappa}{S_\kappa}(t) \cdot \|X\|^2, \quad X \perp \gamma'(t)$$

Where

$$(2.4) \quad S_\kappa(t) = \begin{cases} \sin(\sqrt{\kappa} \cdot t)/\sqrt{\kappa} & \text{if } \kappa > 0 \\ 1/t & \text{if } \kappa = 0 \\ \sinh(\sqrt{-\kappa} \cdot t)/\sqrt{-\kappa} & \text{if } \kappa < 0 \end{cases}$$

and  $C_\kappa(t) = S'_\kappa(t)$ ,

The second and third results we need are eigenvalue estimates proved in [5] and in [4]. The former is a generalization of the well known Barta's eigenvalue theorem [1] and the later is a generalization of a result of Cheung-Leung [10].

**Theorem 2.2** ([5]). *Let  $\Omega$  be an open set in a Riemannian manifold  $M$ . Then*

$$(2.5) \quad \lambda^*(\Omega) \geq \sup_{\mathcal{C}(\Omega)} \{ \inf_{\Omega} (\text{div } X - |X|^2) \}.$$

Where  $\mathcal{C}(\Omega)$  is the set of smooth vector fields in  $\Omega \setminus F$  for some closed set  $F$  with Hausdorff measure  $\mathcal{H}^{n-1}(F \cap \Omega) = 0$ .

**Theorem 2.3** ([4]). *Let  $\Omega$  be an open set in a Riemannian manifold  $M$  and  $c(\Omega)$  a constant defined by*

$$c(\Omega) = \sup_{\mathcal{C}_+(\Omega)} \frac{(\inf_{\Omega} \text{div } X)}{\sup_{\Omega} |X|^2},$$

where  $\mathcal{C}_+(\Omega) = \{X \in \mathcal{C}(\Omega) \mid \inf_{\Omega} \text{div } X > 0 \text{ and } \sup_{\Omega} |X| < \infty\}$ . Then

$$(2.6) \quad \lambda^*(\Omega) \geq \frac{c(\Omega)^2}{4}$$

Finally, we need the following technical lemma.

**Lemma 2.4.** *Let  $v : B_{\mathbb{N}^n(\kappa)}(r) \rightarrow \mathbb{R}$  be a first positive eigenfunction of  $B_{\mathbb{N}^n(\kappa)}(r) \subset \mathbb{N}^n(\kappa)$  associated to the first eigenvalue  $\lambda_1(B_{\mathbb{N}^n(\kappa)}(r))$ . Then*

$$(2.7) \quad n \frac{C_{\kappa}(t)}{S_{\kappa}(t)} v'(t) + \lambda_1(B_{\mathbb{N}^n(\kappa)}(r)) v(t) < 0, \quad t \in (0, r).$$

*Proof.* We are going to treat the cases  $\kappa < 0$ ,  $\kappa = 0$  and  $\kappa > 0$  separately. Suppose first that  $\kappa < 0$  and let us call  $\lambda = \lambda_1(B_{\mathbb{N}^n(\kappa)}(r))$  for simplicity of notation. Recall that  $v(t)$  satisfies the following differential equation.

$$(2.8) \quad v''(t) + (n-1) \frac{C_{\kappa}(t)}{S_{\kappa}(t)} v'(t) + \lambda v(t) = 0, \quad t \in (0, r)$$

Consider the function  $\mu(t) = C_{\kappa}(t) \frac{\lambda}{n\kappa}$ . Thus  $\mu'(t) = -\frac{\lambda}{n} S_{\kappa}(t) C_{\kappa}(t) \frac{\lambda}{n\kappa} - 1$  and

$$(2.9) \quad \begin{aligned} v'(t)\mu(t) - \mu'(t)v(t) &= v'(t) C_{\kappa}(t) \frac{\lambda}{n\kappa} + \frac{\lambda}{n} S_{\kappa}(t) C_{\kappa}(t) \frac{\lambda}{n\kappa} - 1 v(t) \\ &= \frac{1}{n} C_{\kappa}(t) \frac{\lambda}{n\kappa} - 1 S_{\kappa}(t) \left( n \frac{C_{\kappa}(t)}{S_{\kappa}(t)} v'(t) + \lambda v(t) \right) \end{aligned}$$

From (2.9) we see that to prove that  $n \frac{C_{\kappa}(t)}{S_{\kappa}(t)} v'(t) + \lambda v(t) < 0$  we only need to prove

$$v'(t)\mu(t) - \mu'(t)v(t) < 0.$$

Multiplying the equation (2.8) by  $S_{\kappa}^{n-1}$ , we obtain the following differential equation

$$(2.10) \quad (S_{\kappa}^{n-1} v')'(t) + \lambda S_{\kappa}^{n-1}(t) v(t) = 0, \quad t \in (0, r).$$

The function  $\mu(t) = C_{\kappa}(t) \frac{\lambda}{n\kappa}$ , satisfies the differential equation

$$(2.11) \quad \mu''(t) = -\lambda \left( \frac{1}{nC_{\kappa}^2(t)} - \frac{\lambda}{n^2} \frac{S_{\kappa}^2(t)}{C_{\kappa}^2(t)} \right) \mu(t).$$

Multiplying the equation (2.11) by  $S_{\kappa}^{n-1}(t)$  we obtain

$$(2.12) \quad S_{\kappa}^{n-1}(t) \mu''(t) + \lambda S_{\kappa}^{n-1}(t) \left( \frac{1}{nC_{\kappa}^2(t)} - \frac{\lambda}{n^2} \frac{S_{\kappa}^2(t)}{C_{\kappa}^2(t)} \right) \mu(t) = 0.$$

Adding and subtracting the term  $(n-1)\mu'(t)S_{\kappa}^{n-2}(t)C_{\kappa}(t)$  we obtain

$$(2.13) \quad (S_{\kappa}^{n-1}\mu')'(t) + \lambda S_{\kappa}^{n-1}(t) \left( \frac{n-1}{n} + \frac{1}{nC_{\kappa}^2(t)} - \frac{\lambda}{n^2} \frac{S_{\kappa}^2(t)}{C_{\kappa}^2(t)} \right) \mu(t) = 0$$

The functions  $v$  and  $\mu$  then satisfy the follow identities:

$$(2.14) \quad \begin{aligned} (S_{\kappa}^{n-1} v')'(t) + \lambda S_{\kappa}^{n-1}(t) v(t) &= 0 \\ (S_{\kappa}^{n-1}\mu')'(t) + \lambda S_{\kappa}^{n-1}(t) \left( \frac{n-1}{n} + \frac{1}{nC_{\kappa}^2(t)} - \frac{\lambda}{n^2} \frac{S_{\kappa}^2(t)}{C_{\kappa}^2(t)} \right) \mu(t) &= 0 \end{aligned}$$

Multiply the first identity of (2.14) by  $\mu(t)$  and the second identity by  $-v(t)$  adding them and integrating from 0 to  $t$  we obtain

$$(2.15) \quad S_{\kappa}^{n-1}(v'\mu - \mu'v)(t) = - \int_0^t \lambda S_{\kappa}^{n-1}(t) \left( \frac{1}{n} - \frac{1}{nC_{\kappa}^2(t)} + \frac{\lambda}{n^2} \frac{S_{\kappa}^2(t)}{C_{\kappa}^2(t)} \right) \mu(t)v(t) dt$$

Clearly

$$\lambda S_\kappa^{n-1}(t) \left( \frac{1}{n} - \frac{1}{nC_\kappa^2(t)} + \frac{\lambda}{n^2} \frac{S_\kappa^2(t)}{C_\kappa^2(t)} \right) \mu(t)v(t) > 0$$

Therefore we have  $v'(t)\mu(t) - \mu'(t)v(t) < 0$  for  $t \in (0, r)$ . This settle the case  $\kappa < 0$ .

Suppose that  $\kappa > 0$ . We have  $S_\kappa(t) = \frac{1}{\sqrt{\kappa}} \sin \sqrt{\kappa} t$  for  $t \in (0, r)$  with  $r < \frac{\pi}{\sqrt{\kappa}}$ . Define  $\mu(t) = C_\kappa(t) \frac{-\lambda}{n\kappa}$ . Thus  $\mu'(t) = \frac{\lambda}{n} S_\kappa(t) C_\kappa(t) \frac{-\lambda}{n\kappa} - 1$ . With a similar procedure we obtain that  $v$  and  $\mu$  satisfy the following differential identities

$$(2.16) \quad \begin{aligned} (S_\kappa^{n-1} v')'(t) + \lambda S_\kappa^{n-1}(t) v(t) &= 0 \\ (S_\kappa^{n-1} \mu')'(t) - \lambda S_\kappa^{n-1}(t) \left( \frac{n-1}{n} + \frac{1}{nC_\kappa^2(t)} + \frac{\lambda}{n^2} \frac{S_\kappa^2(t)}{C_\kappa^2(t)} \right) \mu(t) &= 0 \end{aligned}$$

In (28) we multiply the first identity by  $\mu$  and the second by  $-v$  adding them and integrating from 0 to  $t$  the resulting identity we obtain

$$(2.17) \quad S_\kappa^{n-1}(v' \mu - \mu'(t)v)(t) = - \int_0^t \lambda_1 S_\kappa^{n-1}(t) \left( 2 - \frac{1}{n} + \frac{1}{nC_\kappa^2(t)} + \frac{\lambda}{n^2} \frac{S_\kappa^2(t)}{C_\kappa^2(t)} \right) \mu(t)v(t) dt$$

The term  $\lambda S_\kappa^{n-1}(t) \left( 2 - \frac{1}{n} + \frac{1}{nC_\kappa^2(t)} + \frac{\lambda}{n^2} \frac{S_\kappa^2(t)}{C_\kappa^2(t)} \right) \mu(t)v(t) > 0$  is positive for  $t \in (0, r)$ ,  $r < \pi/2\sqrt{\kappa}$ . Therefore we have that  $v'(t)\mu(t) - \mu'(t)v(t) < 0$  for  $t \in (0, r)$ ,  $r < \pi/2\sqrt{\kappa}$

The case  $\kappa = 0$  we proceed similarly. Define  $\mu(t) = e^{-\frac{\lambda t^2}{2n}}$ . The functions  $v$  and  $\mu$  satisfy the following identities,

$$(2.18) \quad \begin{aligned} (t^{n-1} v'(t))' + \lambda t^{n-1} v(t) &= 0 \\ (t^{n-1} \mu'(t))' + \lambda t^{n-1} (1 - \frac{\lambda t^2}{n^2}) \mu(t) &= 0 \end{aligned}$$

In (2.18) we multiply the first identity by  $\mu$  and the second by  $-v$  adding them and integrating from 0 to  $t$  the resulting identity we obtain,

$$t^{n-1}(v'(t)\mu(t) - v(t)\mu'(t)) = -\frac{\lambda^2}{n^2} \int_0^t \mu(t)v(t) < 0, \quad \forall t \in (0, r).$$

Then  $\mu(t)v'(t) - \mu'(t)v(t) < 0$ . This proves the lemma. □

### 3. PROOF OF THE RESULTS

#### 3.1. Proof of Theorem 1.1.

*Proof.* Let  $\varphi : M \hookrightarrow N \times \mathbb{R}$  be a minimal immersion of a  $m$ -dimensional Riemannian manifold  $M$ , where  $N$  is a complete Riemannian  $n$ -manifold with radial sectional curvature along the geodesics  $\gamma(t)$  issuing from a point  $x_0 \in N$  bounded from above  $K(\gamma'(t), v) \leq \kappa$ ,  $v \in T_{\gamma(t)}N$ ,  $|v| = 1$ ,  $v \perp \partial t$ . Let  $\Omega \subset \varphi^{-1}(B_N(x_0, r) \times \mathbb{R})$ ,  $r < \min\{\text{inj}(x_0), \pi/2\sqrt{\kappa}\}$ , ( $\pi/2\sqrt{\kappa} = \infty$  if  $\kappa \leq 0$ ) be a connected component. Let  $\rho_N(x) = \text{dist}_N(x_0, x)$  be the distance function in  $N$  to  $x_0$  and let  $v : B_{\mathbb{N}^{m-1}(\kappa)}(r) \rightarrow \mathbb{R}$  be a first positive eigenfunction associated with the first eigenvalue  $\lambda_1(B_{\mathbb{N}^{m-1}(\kappa)}(r))$  of the geodesic ball of radius  $r$  in the simply connected  $(m-1)$ -dimensional space form  $\mathbb{N}^{m-1}(\kappa)$  of constant sectional curvature

$\kappa$ . The eigenfunction  $v$  is radial, i.e.  $v(x) = v(|x|)$  and we can look at  $v$  as it were defined in  $[0, r]$  satisfying the equation

$$(3.1) \quad v''(t) + (m-2) \frac{C_\kappa}{S_\kappa}(t) v'(t) + \lambda_1(B_{\mathbb{N}^{m-1}(\kappa)}(r)) v(t) = 0, \quad t \in (0, r).$$

Choose the first eigenfunction that satisfies the initial conditions  $v(0) = 1$  and  $v'(0) = 0$ . Define  $g : B_N(r) \times \mathbb{R} \rightarrow \mathbb{R}$  by  $g = v \circ \rho_N \circ p$  and  $f : \Omega \rightarrow \mathbb{R}$  by  $f = g \circ \varphi$ , where  $p : N \times \mathbb{R} \rightarrow N$  is the projection in the first factor. Setting  $X = \text{grad} \log f$  we have that  $\text{div} X - |X|^2 = \Delta f / f$ . Thus by Theorem (2.2) we have that

$$\lambda^*(\Omega) \geq \inf_{\Omega} \left( -\frac{\Delta f}{f} \right).$$

We are going to give lower bound  $-\Delta f / f$ . Let  $x \in \Omega$  and  $\{e_1, \dots, e_m\}$  be any orthonormal basis for  $T_x \Omega$ . The Laplacian of  $f$  at  $x$  is given by

$$(3.2) \quad \Delta_M f(x) = \sum_{i=1}^m \text{Hess}_{(N \times \mathbb{R})} g(\varphi(x)) (e_i, e_i) = \sum_{i=1}^m \text{Hess}_N v \circ \rho_N(q) (e_i, e_i)$$

Consider the orthonormal basis  $\{\text{grad} \rho_N, \partial/\partial\theta_1, \dots, \partial/\partial\theta_{n-1}, \partial/\partial s\}$  for  $T_{(q,s)}(N \times \mathbb{R})$ , where  $\{\text{grad} \rho_N, \partial/\partial\theta_1, \dots, \partial/\partial\theta_{n-1}\}$  is an orthonormal basis for  $T_q N$  (polar coordinates). Let  $\{e_1, \dots, e_m\}$  be an orthonormal basis for  $T_x \Omega$  and write

$$(3.3) \quad e_i = a_i \cdot \text{grad} \rho_N + b_i \cdot \partial/\partial s + \sum_{j=1}^{n-1} c_i^j \cdot \partial/\partial\theta_j.$$

Where  $a_i, b_i, c_i^j$  are constants satisfying  $a_i^2 + b_i^2 + \sum_{j=1}^{n-1} (c_i^j)^2 = 1$ ,  $i = 1, \dots, m$ . Computing  $\Delta_M f(x)$  we have, (recall that  $\varphi(x) = (q, s)$  and we are letting  $t = \rho_N(q)$ )

$$\begin{aligned} \Delta f(x) &= \sum_{i=1}^m [e_i(v'(t)) \langle \text{grad} \rho_N, e_i \rangle + v'(t) \text{Hess}_N \rho_N(e_i, e_i)] \\ &= v''(t) \sum_{i=1}^m a_i^2 + v'(t) \sum_{i=1}^m \sum_{j=1}^{n-1} (c_i^j)^2 \text{Hess} \rho_N(\partial/\partial\theta_j, \partial/\partial\theta_j) \end{aligned}$$

Since  $v'(t) \leq 0$  we have by the Hessian Comparison Theorem that

$$\begin{aligned} -\Delta f(x) &\geq -v''(t) \sum_{i=1}^m a_i^2 - v'(t) \frac{C_\kappa}{S_\kappa}(t) \sum_{i=1}^m \sum_{j=1}^{n-1} (c_i^j)^2 \\ &= -v''(t) \sum_{i=1}^m a_i^2 - v'(t) \frac{C_\kappa}{S_\kappa}(t) \left[ m - \sum_{i=1}^m a_i^2 - \sum_{i=1}^m b_i^2 \right] \\ (3.4) \quad &= -v''(t) - (m-2)v'(t) \frac{C_\kappa}{S_\kappa}(t) \\ &\quad + v''(t) \left[ 1 - \sum_{i=1}^m a_i^2 \right] - v'(t) \frac{C_\kappa}{S_\kappa}(t) \left[ 1 - \sum_{i=1}^m a_i^2 + 1 - \sum_{i=1}^m b_i^2 \right] \\ &= \lambda_1(B_{\mathbb{N}^{m-1}(\kappa)}(r)) v(t) \\ &\quad + v''(t) \left[ 1 - \sum_{i=1}^m a_i^2 \right] - v'(t) \frac{C_\kappa}{S_\kappa}(t) \left[ 1 - \sum_{i=1}^m a_i^2 + 1 - \sum_{i=1}^m b_i^2 \right] \end{aligned}$$

We will show that the last line of (3.4) is nonnegative, this is

$$(3.5) \quad v''(t) \left[ 1 - \sum_{i=1}^m a_i^2 \right] - v'(t) \frac{C_\kappa}{S_\kappa}(t) \left[ 1 - \sum_{i=1}^m a_i^2 + 1 - \sum_{i=1}^m b_i^2 \right] \geq 0.$$

Substituting  $v''(t) = -(m-2)v'(t) \frac{C_\kappa}{S_\kappa}(t) - \lambda_1(B_{\mathbb{N}^{m-1}(\kappa)}(r))v(t)$  in (3.5) we obtain

$$(3.6) \quad \begin{aligned} & v''(t) \left[ 1 - \sum_{i=1}^m a_i^2 \right] - v'(t) \frac{C_\kappa}{S_\kappa}(t) \left[ 1 - \sum_{i=1}^m a_i^2 + 1 - \sum_{i=1}^m b_i^2 \right] = \\ & - \left[ (m-1)v'(t) \frac{C_\kappa}{S_\kappa}(t) + \lambda_1(B_{\mathbb{N}^{m-1}(\kappa)}(r))v(t) \right] \left[ 1 - \sum_{i=1}^m a_i^2 \right] \\ & - v'(t) \frac{C_\kappa}{S_\kappa}(t) \left[ 1 - \sum_{i=1}^m b_i^2 \right] \geq 0 \end{aligned}$$

since we have that  $(m-1)v'(t) \frac{C_\kappa}{S_\kappa}(t) + \lambda_1(B_{\mathbb{N}^{m-1}(\kappa)}(r))v(t) < 0$  by Lemma (2.4) and  $[1 - \sum_{i=1}^m a_i^2] \geq 0$  and  $[1 - \sum_{i=1}^m b_i^2] \geq 0$ . From (3.4) we have  $-\frac{\Delta f}{f}(x) \geq \lambda_1(B_{\mathbb{N}^{m-1}(\kappa)}(r))$ . Therefore,

$$\lambda^*(\Omega) \geq \inf_{\Omega} (-\Delta f/f) \geq \lambda_1(B_{\mathbb{N}^{m-1}(\kappa)}(r)).$$

To prove the last assertion of Theorem (1.1) we need the following proposition proved in [5].

**Proposition 3.1.** *Let  $\Omega$  be a bounded domain in a smooth Riemannian manifold. Let  $v \in C^2(\Omega) \cap C^0(\overline{\Omega})$ ,  $v > 0$  in  $\Omega$  and  $v|_{\partial\Omega} = 0$ . Then*

$$(3.7) \quad \lambda^*(\Omega) \geq \inf_{\Omega} \left( -\frac{\Delta v}{v} \right).$$

Moreover,  $\lambda^*(\Omega) = \inf_{\Omega} \left( -\frac{\Delta v}{v} \right)$  if and only if  $v = u$ , where  $u$  is a positive eigenfunction of  $\Omega$ , i.e.  $\Delta u + \lambda^*(\Omega)u = 0$ .

If we have equality  $\lambda_1(\Omega) = \lambda_1(B_{\mathbb{N}^{m-1}(\kappa)}(r))$  we have that  $f$  is an eigenfunction and the expression (3.6) is zero (at each point of  $\Omega$ ). This happens if and only if

$$1 = \sum_{i=1}^m \alpha_i^2 = \sum_{i=1}^m \beta_i^2.$$

On the other hand, we can write at each point  $x \in \Omega$

$$\text{grad}_N = \sum_{i=1}^m \alpha_i e_i + (\text{grad}_N)^\perp,$$

where  $(\text{grad}_N)^\perp$  is normal to the tangent space of  $T_x\Omega$ . Likewise we can write

$$\partial/\partial s = \sum_{i=1}^m \beta_i e_i + (\partial/\partial s)^\perp.$$

Since  $\|\text{grad}_N\|^2 = \sum_{i=1}^m \alpha_i^2 + \|(\text{grad}_N)^\perp\|^2$  and  $\|\partial/\partial s\|^2 = \sum_{i=1}^m \beta_i^2 + \|(\partial/\partial s)^\perp\|^2$  we conclude that  $(\text{grad}_N)^\perp = 0 = (\partial/\partial s)^\perp$ . Thus the tangent space  $T_x\Omega$  contains the vectors  $\text{grad}_N$  and  $\partial/\partial s$  for each  $x \in \Omega$ . Thus, we could have chosen in (??) an orthonormal basis for  $T_x\Omega$  in the following way  $e_1 = \text{grad}_N$ ,  $e_2 = \partial/\partial s$  and  $\{e_3, \dots, e_m\} \subset \{\partial/\partial\theta_1, \dots, \partial/\partial\theta_{n-1}\}$ . Clearly the set of vectors  $\text{grad}_N$  and  $\partial/\partial s$  form smooth vector fields on  $\Omega$  since they are the restrictions of smooth vector fields on  $N \times \mathbb{R}$  to a smooth immersed submanifold. The integral

curves of the vector field  $\partial/\partial s$  in  $\Omega$  are  $\{x\} \times \mathbb{R}$  contained in  $\varphi(M)$  and  $\Omega = \varphi^{-1}(B_N(r) \times \mathbb{R})$  is not bounded. This proves Theorem (1.1).  $\square$

**3.2. Proof of Theorem 1.6.** Let  $\varphi : M \hookrightarrow N \times \mathbb{R}$  be a complete immersed  $m$ -submanifold with locally bounded mean curvature, where  $N$  has radial sectional curvature bounded above  $K_N \leq \kappa$  along the geodesics issuing from  $x_0$ . Define  $\tilde{\rho}_N : N \times \mathbb{R} \rightarrow \mathbb{R}$  by  $\tilde{\rho}_N(x, t) = \rho_N(x)$ ,  $\rho_N(x) = \text{dist}_N(x_0, x)$ . Let  $\Omega(r) = \varphi^{-1}(B_N(x_0, r) \times \mathbb{R})$ ,  $f = \tilde{\rho}_N \circ \varphi : \Omega(r) \rightarrow \mathbb{R}$  and  $X = \text{grad } f$ . The idea is to choose  $r < \min\{\text{inj}_N(x_0), \pi/2\sqrt{\kappa}\}$ ,  $\pi/2\sqrt{\kappa} = \infty$  if  $\kappa \leq 0$ , properly such that  $\inf_{\Omega(r)} \text{div } X > 0$  then by Theorem (2.3) we have that

$$\lambda^*(\Omega(r)) \geq \left( \frac{\inf \text{div } X}{2 \sup |X|} \right)^2.$$

Observe that  $\text{div } X = \Delta_M f$  and as in (2.2) we have

$$\Delta_M f(x) = \left[ \sum_{i=1}^m \text{Hess}_{N \times \mathbb{R}} \tilde{\rho}_N(e_i, e_i) + \langle \text{grad}_{N \times \mathbb{R}} \tilde{\rho}_N, \vec{H} \rangle \right] (\varphi(x))$$

Where  $\vec{H} = \sum_{i=1}^m \alpha(e_i, e_i)$  is the mean curvature vector of  $\varphi(M)$  at  $\varphi(x)$  and  $\{e_1, \dots, e_m\}$  is an orthonormal basis of  $T_x M$  as in (3.3) identified with  $\{d\varphi \cdot e_1, \dots, d\varphi \cdot e_m\}$ . Now

$$\begin{aligned} \sum_{i=1}^m \text{Hess}_{N \times \mathbb{R}} \tilde{\rho}_N(e_i, e_i) &= \sum_{i=1}^m \text{Hess}_N \rho_N(e_i, e_i) \\ (3.8) \quad &= \sum_{i=1}^m \sum_{j=1}^{n-1} (c_j^i)^2 \text{Hess}_N \rho_N(\partial/\partial \theta_j, \partial/\partial \theta_j) \\ &\geq \sum_{i=1}^m (1 - a_i^2 - b_i^2) \frac{C_\kappa}{S_\kappa}(r) \end{aligned}$$

On the other hand  $\langle \text{grad}_{N \times \mathbb{R}} \tilde{\rho}_N, \vec{H} \rangle = \langle \text{grad}_N \rho_N, \vec{H} \rangle$

$$\begin{aligned} \langle \text{grad}_N \rho_N, \vec{H} \rangle &= \langle (\text{grad}_N \rho_N)^\perp, \vec{H} \rangle \\ (3.9) \quad &\leq |H| \sqrt{1 - \sum_{i=1}^m a_i^2} \\ &\leq h(x_0, r) \sqrt{1 - \sum_{i=1}^m a_i^2} \end{aligned}$$

Since  $|\text{grad}_N \rho_N|^\perp|^2 = (1 - \sum_{i=1}^m a_i^2)$ . Therefore from (3.9) and (3.9) we have

$$\Delta_M f(x) \geq (m-2) \frac{C_\kappa}{S_\kappa}(r) - h(x_0, r) > 0$$

We have two cases to consider. First is the case that  $|h(x_0, r)| < \Lambda^2 < \infty$  and we choose  $r \leq \min\{\text{inj}_N(x_0), \pi/2\sqrt{\kappa}, (C_\kappa/S_\kappa)^{-1}(\Lambda^2/(m-2))\}$ . In case that  $\lim_{r \rightarrow \infty} h(x_0, r) = \infty$  there is  $r_0$  so that  $(m-2) \frac{C_\kappa}{S_\kappa}(r_0) - h(x_0, r_0) = 0$  since we can assume without loss of generality that  $h(x_0, r)$  is a continuous non-decreasing function in  $r$ . Then we choose  $r \leq \min\{\text{inj}_N(x_0), \pi/2\sqrt{\kappa}, (C_\kappa/S_\kappa)^{-1}(h(x_0, r_0)/(m-2))\}$ . In both cases we have

$$\lambda^*(\Omega(r)) \geq \left[ \frac{(m-2) \frac{C_\kappa}{S_\kappa}(r) - h(x_0, r)}{2} \right]^2.$$

To prove Corollary (1.7) just see that  $\Omega(R_{p_1(M)}) = M$  and  $\lambda^*(M) = 0$ .



## REFERENCES

1. J. Barta, *Sur la vibration fondamentale d'une membrane*. C. R. Acad. Sci. **204**, (1937), 472–473.
2. Bérard, Pierre H. *Spectral geometry: direct and inverse problems*. Lect. Notes in Math. 1207, 1986, Springer-Verlag.
3. Berger, M., Gauduchon, P. and Mazet, E.: *Le Spectre d'une Variété Riemannienne*. Lect. Notes Math. 194, 1974, Springer-Verlag.
4. G. P. Bessa and J. F. Montenegro, *Eigenvalue estimates for submanifolds with locally bounded mean curvature*. Ann. Global Anal. and Geom., **24**, (2003), 279–290.
5. G. P. Bessa, and J. Fabio Montenegro, *An Extension of Barta's Theorem and Geometric Applications*. Ann. Global Anal. Geom. **31** (2007), no. 4, 345–362.
6. G. P. Bessa, and J. Fabio Montenegro, *On compact H-hypersurfaces of  $N \times \mathbb{R}$* . Geom. Dedicata **127**, (2007), 1–5.
7. A. Candel, *Eigenvalue estimates for minimal surfaces in Hyperbolic space*. Trans. Amer. Math. Soc. **359**, (2007), 3567–3575.
8. I. Chavel, *Eigenvalues in Riemannian Geometry*. Pure and Applied Mathematics, 1984, Academic Press, INC.
9. S. Y. Cheng, P. Li, S. T. Yau, *Heat equations on minimal submanifolds and their applications*. Amer. J. Math. **106**, 1033–1065, (1984).
10. Leung-Fu Cheung and Pui-Fai Leung, *Eigenvalue estimates for submanifolds with bounded mean curvature in the hyperbolic space*. Math. Z. **236**, (2001), 525–530.
11. L. Jorge, and D. Koutrofiotis, *An estimate for the curvature of bounded submanifolds*. Amer. J. Math., **103**, (1980), 711–725.
12. L. Jorge, and F. Xavier, *A complete minimal surface in  $\mathbb{R}^3$  between two parallel planes*. Ann. of Math. (2) **112** (1980), no. 1, 203–206.
13. F. Martín, S. Morales, *A complete bounded minimal cylinder in  $\mathbb{R}^3$* . Michigan Math. J. **47** (2000), no. 3, 499–514.
14. W. Meeks and H. Rosenberg, *The theory of minimal surfaces in  $M^2 \times \mathbb{R}$* . Comment. Math. Helv. **80** (2005), no. 4, 811–858.
15. W. Meeks and H. Rosenberg, *Stable minimal surfaces in  $M^2 \times \mathbb{R}$* . J. Differential Geom. **68** (2004), no. 3, 515–534.
16. N. Nadirashvili, *Hadamard's and Calabi-Yau's conjectures on negatively curved and minimal surfaces*. Invent. Math., **126**, (1996), 457–465.
17. R. Schoen, and S. T. Yau, *Lectures on Differential Geometry*. Conference Proceedings and Lecture Notes in Geometry and Topology, **vol. 1**, (1994).

THE ABDUS SALAM INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS, 34014 TRIESTE, ITALY

*Current address:* Department of Mathematics, Universidade Federal do Ceara-UFC, Campus do Pici, 60455-760 Fortaleza-CE Brazil

*E-mail address:* `bessa@mat.ufc.br`

DEPARTMENT OF ENGINEERING, UNIVERSIDADE FEDERAL DO CEARA-UFC, CAMPUS CARIRI, AV. CASTELO BRANCO, 150, 60030-200 JUAZEIRO DO NORTE-CE, BRAZIL

*E-mail address:* `silvana_math@yahoo.com.br`